Math 3236 Statistical Theory

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1 2 3

p_1 p_2 p_3 $p_1 + p_2 + p_3 = 1$

X_i random sample

with p.m.f.

$$p(i \mid \underline{p}) = p_i$$

Bayes MBD Distribution

Dirichlet "

$$g(\underline{p} \mid \underline{\alpha}) = \frac{1}{B(\underline{\alpha})} p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} p_3^{\alpha_3 - 1}$$

$$B(\underline{\alpha}) = \int_{p_1 + p_2 < 1} p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} (1 - p_1 - p_2)^{\alpha_3 - 1} \, dp_1 \, dp_2$$

$$= \int_0^1 p_1^{\alpha_1 - 1} \left(\int_0^{1-p_1} p_2^{\alpha_2 - 1} (1 - p_1 - p_2)^{\alpha_3 - 1} \, dp_2 \right) dp_1$$

$$y_1 = p_1 \qquad y_2 = \frac{p_2}{1 - p_1}$$

$$B(\underline{\alpha}) = \int_0^1 y_1^{\alpha_1 - 1} (1 - y_1)^{\alpha_2 + \alpha_3 - 1}$$

$$\int_0^1 y_2^{\alpha_2 - 1} (1 - y_2)^{\alpha_3 - 1} \, dy_2 \, dy_1$$

$$=$$

$$(1 - p_1 - p_2) = (1 - y_1)(1 - y_2)$$

$$= B(\underline{\alpha}) = B(\alpha_1, \alpha_2 + \alpha_3) \, B(\alpha_2, \alpha_3)$$

$$= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}$$

$n_k(\underline{x})$ = number of $x_i = k$

$$g(p) \sim p_1^{\alpha_1 - 1} \, p_2^{\alpha_2 - 1} \, p_3^{\alpha_3 - 1}$$

$$g(p \mid \underline{x}) = p_1^{\alpha_1 - 1} \, p_2^{\alpha - 1} \, p_3^{\alpha_3 - 1} \, p_1^{n_1(\underline{x})} \, p_2^{n_2(\underline{x})} \, p_3^{n_3(x)}$$

$g(p \mid \underline{x})$ is a MBD with

$$\alpha_1' = \alpha_1 + n_1(\underline{x})$$

$$\alpha_2' = \alpha_2 + n_2(\underline{x})$$

$$\alpha_3' = \alpha_3 + n_3(\underline{x})$$

$$\mathbb{E}\left(c_1 (p_1 - a_1)^2 + c_2 (p_2 - a_2)^2 + c_3 (p_3 - a_3)^2 \mid \underline{X} \right)$$

$$a_1 + a_2 + a_3 = 1$$

$$c_1 \, \mathbb{E}(p_1 - a_1 \mid \underline{X}) = \lambda$$

$$c_2 \, \mathbb{E}(p_2 - a_2 \mid \underline{X}) = \lambda$$

$$c_3 \, \mathbb{E}(p_3 - a_3 \mid X) = \lambda$$

$$a_k = \mathbb{E}(p_k \mid X) - \lambda / c_k$$

$$\sum \mathbb{E}(p_k \mid X) = 1$$

$$\lambda = 0 \quad !$$

$$a_k = \mathbb{E}(p_k \mid X)$$

$$\mathbb{E}_{\underline{\alpha}}(p_1) = \frac{1}{B(\underline{\alpha})} \int_{p_1+p_2<1} p_1 \, p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} (1 - p_1 - p_2)^{\alpha_3 - 1} \, dp_1 dp_2$$

$$= \frac{B(\alpha_1 + 1, \alpha_2, \alpha_3)}{B(\alpha_1, \alpha_2, \alpha_3)} = \frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3}$$

$$\hat{p}_k^{B}(\underline{X}) = \frac{\alpha_k + n_k(\underline{X})}{N + \sum_k \alpha_k}$$

$$p_1 \quad p_2 \quad p_3$$

$$\frac{n_k(\underline{X})}{N} \xrightarrow{P} p_k$$

$$\Downarrow$$

$$\hat{p}_k^B(\underline{X}) \xrightarrow{P} p_k$$

$$\mathbb{P}\left(\sum_{k=1}^{3} |\hat{p}_k^B(X) - p_k| > \delta\right) \leq$$

$$\sum \mathbb{P}\left(|\hat{p}_k^B(\underline{X}) - p_k| > \delta/3\right)$$

$$L(\underline{p} \mid \underline{x}) = p_1^{n_1(\underline{x})} p_2^{n_2(\underline{x})} p_3^{n_3(\underline{x})}$$

$$\ell(\underline{p} \mid \underline{x}) = n_1(\underline{x}) \ln p_1 + n_2(\underline{x}) \ln p_2 + n_3(\underline{x}) \ln p_3$$

$$\frac{n_k(x)}{p_k} = \lambda$$

$$\hat{p}_k^L(\underline{x}) = \frac{n_k(\underline{X})}{N}$$

$\alpha_1 = \alpha_2 = \alpha_3 = 0$ improper prior.

$$Y_i \quad \text{r.v.} \quad \begin{cases} 1 & \text{if } X_i = 1 \\ 0 & \text{if } X_i \neq 1 \end{cases}$$

$$n_1(X) = \sum_{i=1}^{N} Y_i$$

$$\mathbb{E}(\hat{p}_1^L) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}(Y_i) = p_1$$

$$Var(\hat{p}_1^L) = \frac{p_1(1-p_1)}{N}$$

$$Z_i \quad \text{r.v.} \quad \begin{cases} 1 & \text{if } X_i = 2 \\ 0 & \text{if } X_i \neq 2 \end{cases}$$

$$cov(\hat{p}_1^L, \hat{p}_2^L) = \frac{1}{N^2} \sum_{i=1}^{N} cov(Y_i, Z_i) =$$

$$= -\frac{p_1 p_2}{N}$$

$$\sqrt{N}(\hat{p}_1^L - p_1, \hat{p}_2^L - p_2) \Rightarrow (Z_1, Z_2)$$

$$f(z_1, z_2) = \frac{1}{2\pi\sqrt{p_1 p_2 p_3 (1-p_1)(1-p_2)}}$$

$$\exp\left(-\frac{1}{2p_3}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}' \begin{pmatrix} \frac{1}{p_1(1-p_1)} & \frac{1}{\sqrt{p_1 p_2 (1-p_1)(1-p_2)}} \\ 0 & \frac{1}{p_2(1-p_2)} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$$

$$C = \begin{pmatrix} p_1(1-p_1) & -p_1 p_2 \\ -p_1 p_2 & p_2(1-p_2) \end{pmatrix}$$

$$\exp\left(\frac{1}{2}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}' C^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$$

2)

$$\alpha' = \alpha + NK \qquad \beta' = \beta + \sum_i x_i$$

$$\mathbb{E}\left[(\lambda - a)^4\right]$$

$$4\,\mathbb{E}\left[(\lambda - a)^3\right] = 0$$

$$0 = a^3 - 3\frac{\alpha}{\beta}a^2 + 3\frac{\alpha(\alpha+1)}{\beta^2}a - \frac{\alpha(\alpha+1)(\alpha+2)}{\beta^3}$$

$$\frac{\alpha}{\beta} \downarrow \lambda \qquad \frac{\alpha(\alpha+1)}{\beta^2} \downarrow \lambda^2 \qquad \lambda^3$$

$$a^3 - 3a\lambda + 3a\lambda^2 - \lambda^3 = (a - \lambda)^3 = 0$$

$$\frac{\alpha}{\beta} = \frac{NK}{\sum_i X_i} \longrightarrow \lambda$$

$$3a^2 - 6\frac{\alpha}{\beta}a + 3\frac{\alpha(\alpha+1)}{\beta^2}$$

$$3\left(a - \frac{\alpha}{\beta}\right)^2 + 3\frac{\alpha}{\beta^2}$$

$$f(X, a) = 0$$

$$f'(X, a) > 0$$

$$f'(X, a) > \frac{3a}{\beta^2}$$

x	$\frac{\alpha}{\beta}$	$\frac{\alpha(\alpha+1)}{\beta^2}$	$\frac{\alpha(\alpha+1)(\alpha+2)}{\beta^3}$
	λ	λ^2	λ^3
	$+$	$+$	$+$
	δ	δ	δ

$$a^3 - 3a^2(\lambda + \delta_1) + 3(\lambda^2 + \delta_2)a + (\lambda^3 + \delta_3) = 0$$

$a = \lambda$ unique solution for

$\delta_1 = \delta_2 = \delta_3 = 0$

$a(\delta_1, \delta_2, \delta_3)$

$a(0, 0, 0) = \lambda$